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A note on slowly decaying solutions of the defocusing nonlinear Schrödinger equation

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Abstract. In this paper we obtain and discuss a class of slowly decaying oscillatory solutions of the defocusing NLS. We also show that these solutions exhibit nonlinear interference, i.e. they can be combined to obtain rather general profiles and wavepackets on an arbitrarily given interval.

In the last few years there have been several papers published [Mat1, Mat2, Kov1, Kov2, Sta1, Beu1] on a class of slowly decaying solutions of the Korteweg–de Vries (KdV) and sine–Gordon equations. These solutions exhibit new, rather interesting properties that exponentially decaying solutions do not possess. It seemed only natural to us to consider analogues of these solutions for the nonlinear Schrödinger equation (NLS) [Abl1]

$$\mathrm{i}q_t + q_{xx} + 2\nu|q|^2q = 0 \qquad \nu = \pm 1. \tag{1}$$

It turns out that the method for computing such solutions does not work for the self-focusing NLS ($\nu = 1$), so either the solutions do not exist or a different approach is required to compute them. For the defocusing NLS ($\nu = -1$), however, the corresponding solutions can be constructed as superpositions of slowly decaying singular solitons, which, by analogy with the solutions of [Kov1], we call harmonic solitons. To actually compute such solutions either the method of Darboux transform as developed by Matveev in [Mat3] or the method of [Neu1], could be employed. We use the latter.

To do the actual construction, we first recall that the NLS can be written as a compatibility condition for:

$$\frac{\partial\Phi}{\partial x} = \begin{pmatrix} \mathrm{i}\lambda & q \\ r & -\mathrm{i}\lambda \end{pmatrix}\Phi \qquad \frac{\partial\Phi}{\partial t} = \begin{pmatrix} 2\mathrm{i}\lambda^2 - \mathrm{i}rq & 2\mathrm{i}\lambda q + q_x \\ 2\mathrm{i}\lambda r - r_x & -2\mathrm{i}\lambda^2 + \mathrm{i}rq \end{pmatrix}\Phi$$

$r = -\bar{q}$.

For $q = r = 0$ and $\nu = -1$

$$\Phi_0 = \begin{pmatrix} \varphi_{01}(x,t,\lambda) & \psi_{01}(x,t,\lambda) \\ \varphi_{02}(x,t,\lambda) & \psi_{02}(x,t,\lambda) \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\lambda x + 2\mathrm{i}\lambda^2 t} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\lambda x - 2\mathrm{i}\lambda^2 t} \end{pmatrix}.$$

The formulae of [Neu1] then give us n-soliton solutions of the defocusing NLS in the form:

$$q = -2\mathrm{i}\frac{\begin{vmatrix} 1 & \beta_1 & \lambda_1 & \lambda_1\beta_1 & \lambda_1^2 & \lambda_1^2\beta_1 & \cdots & \lambda_1^{n-1} & \lambda_1^n \\ 1 & \beta_2 & \lambda_2 & \lambda_2\beta_2 & \lambda_2^2 & \lambda_2^2\beta_2 & \cdots & \lambda_2^{n-1} & \lambda_2^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \beta_{2n} & \lambda_{2n} & \lambda_{2n}\beta_{2n} & \lambda_{2n}^2 & \lambda_{2n}^2\beta_{2n} & \cdots & \lambda_{2n}^{n-1} & \lambda_{2n}^n \end{vmatrix}}{\begin{vmatrix} 1 & \beta_1 & \lambda_1 & \lambda_1\beta_1 & \lambda_1^2 & \lambda_1^2\beta_1 & \cdots & \lambda_1^{n-1} & \lambda_1^{n-1}\beta_1 \\ 1 & \beta_2 & \lambda_2 & \lambda_2\beta_2 & \lambda_2^2 & \lambda_2^2\beta_2 & \cdots & \lambda_2^{n-1} & \lambda_2^{n-1}\beta_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \beta_{2n} & \lambda_{2n} & \lambda_{2n}\beta_{2n} & \lambda_{2n}^2 & \lambda_{2n}^2\beta_{2n} & \cdots & \lambda_{2n}^{n-1} & \lambda_{2n}^{n-1}\beta_{2n} \end{vmatrix}} \tag{2}$$

where $\beta_j = \text{constant} \times \frac{\varphi_{01}(x,t,\lambda_j)}{\psi_{02}(x,t,\lambda_j)} = \prod_{\substack{k=1\\k\neq j}}^{n} \frac{\lambda_j-\lambda_k}{\lambda_j-\bar{\lambda}_k} \mathrm{e}^{2(\mathrm{i}\gamma_j\xi_j+p_j\eta_j+\mathrm{i}\lambda_j x+2\mathrm{i}\lambda_j^2 t)}$, $\beta_{n+j} = \bar{\beta}_j^{-1} = \prod_{\substack{k=1\\k\neq j}}^{n} \frac{\bar{\lambda}_j-\lambda_k}{\bar{\lambda}_j-\bar{\lambda}_k} \mathrm{e}^{2(\mathrm{i}\gamma_j\xi_j-p_j\eta_j+\mathrm{i}\lambda_{n+j}x+2\mathrm{i}\lambda_{n+j}^2 t)}$, $\lambda_j = \xi_j + \mathrm{i}\eta_j$, $\lambda_{n+j} = \xi_j - \mathrm{i}\eta_j$ and γ_j, ξ_j, p_j, η_j are some constants, $j \leqslant n$.

Slowly decaying solitons appear when η_j vanish, in which case both the numerator and the denominator of (2) are equal to zero. To solve that problem, we need to take the limit of (2) as $\eta_j \to 0$, which yields:

$$q = -2\mathrm{i}\frac{\begin{vmatrix} 1 & \beta_1 & \xi_1 & \xi_1\beta_1 & \xi_1^2 & \xi_1^2\beta_1 & \cdots & \xi_1^{n-1} & \xi_1^n \\ 1 & \beta_2 & \xi_2 & \xi_2\beta_2 & \xi_2^2 & \xi_2^2\beta_2 & \cdots & \xi_2^{n-1} & \xi_2^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \beta_n & \xi_n & \xi_n\beta_n & \xi_n^2 & \xi_n^2\beta_n & \cdots & \xi_n^{n-1} & \xi_n^n \\ 0 & \mu_1\beta_1 & 1 & (1+2\mu_1\xi_1)\beta_1 & 2\xi_1 & (2\xi_1+\mu_1\xi_1^2)\beta_1 & \cdots & (n-1)\xi_1^{n-2} & n\xi_1^{n-1} \\ 0 & \mu_2\beta_2 & 1 & (1+2\mu_2\xi_2)\beta_2 & 2\xi_2 & (2\xi_2+\mu_2\xi_2^2)\beta_2 & \cdots & (n-1)\xi_2^{n-2} & n\xi_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \mu_n\beta_n & 1 & (1+2\mu_n\xi_n)\beta_n & 2\xi_n & (2\xi_n+\mu_n\xi_n^2)\beta_n & \cdots & (n-1)\xi_n^{n-2} & n\xi_n^{n-1} \end{vmatrix}}{\begin{vmatrix} 1 & \beta_1 & \xi_1 & \xi_1\beta_1 & \xi_1^2 & \xi_1^2\beta_1 & \cdots & \xi_1^{n-1} & \xi_1^{n-1}\beta_1 \\ 1 & \beta_2 & \xi_2 & \xi_2\beta_2 & \xi_2^2 & \xi_2^2\beta_2 & \cdots & \xi_2^{n-1} & \xi_2^{n-1}\beta_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \beta_n & \xi_n & \xi_n\beta_n & \xi_n^2 & \xi_n^2\beta_n & \cdots & \xi_n^{n-1} & \xi_n^{n-1}\beta_n \\ 0 & \mu_1\beta_1 & 1 & (1+\mu_1\xi_1)\beta_1 & 2\xi_1 & (2+\mu_1\xi_1)\xi_1\beta_1 & \cdots & (n-1)\xi_1^{n-2} & ((n-1)+\mu_1\xi_1)\xi_1^{n-2}\beta_1 \\ 0 & \mu_2\beta_2 & 1 & (1+\mu_2\xi_2)\beta_2 & 2\xi_2 & (2+\mu_2\xi_2)\xi_2\beta_2 & \cdots & (n-1)\xi_2^{n-2} & ((n-1)+\mu_2\xi_2)\xi_2^{n-2}\beta_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \mu_n\beta_n & 1 & (1+\mu_n\xi_n)\beta_n & 2\xi_n & (2+\mu_n\xi_n)\xi_n\beta_n & \cdots & (n-1)\xi_n^{n-2} & ((n-1)+\mu_n\xi_n)\xi_n^{n-2}\beta_n \end{vmatrix}} \tag{3}$$

where $\beta_j = \mathrm{e}^{2\mathrm{i}\xi_j(\gamma_j+x+2\xi_j t)}$, $\mu_j = -2\mathrm{i}(p_j - x - 4\xi_j t)$, γ_j, ξ_j, p_j are some constants, $j \leqslant n$. The $(n+j)$th row of (3) for either the numerator or denominator is obtained by subtracting the jth row of (2) from the $(n+j)$th row of (2), dividing the result by η_j and taking its limit as all $\eta_j \to 0$, while for simplicity's sake we may assume that all η_j are equal.

In the simplest case of $n = 1$, (3) gives us

$$\begin{aligned} q(x,t) &= \lim_{\eta\to 0} \frac{2\eta\ \exp(-2\mathrm{i}(\xi x + 2(\xi^2-\eta^2)t + \gamma\xi))}{\sinh(2\eta(x - p + 4\xi t))} = -2\mathrm{i}\frac{\begin{vmatrix} 1 & \xi \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & \beta \\ 0 & \mu\beta \end{vmatrix}} \\ &= \frac{\exp(-2\mathrm{i}\xi(x + 2\xi t + \gamma)}{x - p + 4\xi t} \end{aligned} \tag{4}$$

which we call a single-harmonic soliton solution of the defocusing NLS.

An appropriate modification of the formulae of [Neu1] also allows us to compute

$$\Phi = \lim_{\eta\to 0} \frac{1}{2\sinh(\theta)} \begin{pmatrix} [(\lambda-\xi-\mathrm{i}\eta)\mathrm{e}^{\theta} + (-\mathrm{i}\eta+\xi-\lambda)\mathrm{e}^{-\theta}]\mathrm{e}^{-\mathrm{i}\lambda x-2\mathrm{i}\lambda^2 t} & 2\mathrm{i}\eta \mathrm{e}^{\mathrm{i}[x(\lambda-2\xi)+2t(2\eta^2-2\xi^2+\lambda^2)-2\gamma\xi]} \\ -2\mathrm{i}\eta \mathrm{e}^{-\mathrm{i}[x(\lambda-2\xi)+2t(2\eta^2-2\xi^2+\lambda^2)-2\gamma\xi]} & [(\lambda-\xi+\mathrm{i}\eta)\mathrm{e}^{\theta} + (\mathrm{i}\eta+\xi-\lambda)\mathrm{e}^{-\theta}]\mathrm{e}^{\mathrm{i}\lambda x+2\mathrm{i}\lambda^2 t} \end{pmatrix}$$

$$= \frac{1}{x-p+4\xi t} \begin{pmatrix} [\mathrm{i}+2(\lambda-\xi)(p-x-4t\xi)]\exp(-\mathrm{i}\lambda(x+2\lambda t)) & -\mathrm{i}\exp(\mathrm{i}[-2\gamma\xi-4t\xi^2-2x\xi+\lambda x+2\lambda^2 t]) \\ \mathrm{i}\exp(\mathrm{i}[2\gamma\xi+4t\xi^2+2x\xi-\lambda x-2\lambda^2 t]) & [-\mathrm{i}+2(\lambda-\xi)(p-x-4t\xi)]\exp(\mathrm{i}\lambda(x+2\lambda t)) \end{pmatrix} \tag{5}$$

with $\theta = 2\eta(x-p+4t\xi)$.

Applying (3) with $n = 2$ we obtain a two-harmonic soliton solution of the defocusing NLS:

$$q(x,t) = -2\mathrm{i} \begin{vmatrix} 1 & \beta_1 & \xi_1 & \xi_1^2 \\ 1 & \beta_2 & \xi_2 & \xi_2^2 \\ 0 & \mu_1\beta_1 & 1 & 2\xi_1 \\ 0 & \mu_2\beta_2 & 1 & 2\xi_2 \end{vmatrix} \begin{vmatrix} 1 & \beta_1 & \xi_1 & \xi_1\beta_1 \\ 1 & \beta_2 & \xi_2 & \xi_2\beta_2 \\ 0 & \mu_1\beta_1 & 1 & (1+\mu_1\xi_1)\beta_1 \\ 0 & \mu_2\beta_2 & 1 & (1+\mu_2\xi_2)\beta_2 \end{vmatrix}^{-1}$$

$$= \frac{(\xi_1-\xi_2)}{\sin^2[\gamma_1\xi_1-\gamma_2\xi_2+t(\xi_1^2-\xi_2^2)+x(\xi_1-\xi_2)]+(\xi_1-\xi_2)^2\tau_1\tau_2} \times[\mathrm{e}^{-2\mathrm{i}\xi_1(x+\xi_1 t+\gamma_1)}(\mathrm{i}+(\xi_1-\xi_2)\tau_2)+\mathrm{e}^{-2\mathrm{i}\xi_2(x+\xi_2 t+\gamma_2)}(-\mathrm{i}+(\xi_1-\xi_2)\tau_1)] \tag{6a}$$

where

$$\tau_k = p_k - x - 2t\xi_k \qquad k = 1, 2. \tag{6b}$$

Note that formulae (6) only make sense when $\xi_2 \neq \xi_1$. For $\gamma_2\xi_2 = \gamma_1\xi_1 + m\pi$, where m is an integer number, the concept of superposition of two harmonic solitons can be naturally extended to the case $\xi_2 = \xi_1$ by taking limit of (6) as $\xi_2 \longrightarrow \xi_1$. This yields a harmonic soliton with $\xi = \xi_1 = \xi_2$, $\gamma = \gamma_1 = \gamma_2$ and p given by the equation

$$\frac{1}{p+\frac{\partial}{\partial\xi_2}(\gamma_2\xi_2)} = \frac{1}{p_1+\frac{\partial}{\partial\xi_2}(\gamma_2\xi_2)} + \frac{1}{p_2+\frac{\partial}{\partial\xi_2}(\gamma_2\xi_2)}. \tag{7}$$

For large p_j we can neglect the $\frac{\partial}{\partial\xi_2}(\gamma_2\xi_2)$ term and that will give us

$$\frac{1}{p} \approx \frac{1}{p_1} + \frac{1}{p_2}. \tag{8}$$

Formulae (7) and (8) are exactly the same as the corresponding formulae for KdV [Kov1, Kov2] and although it was expected for an approximate formula (8), the exact similarity of (7) to its KdV counterpart was rather unexpected.

In a finite region $\boldsymbol{D} = \{x, t \mid |x| \leqslant X,\ 0 \leqslant t \leqslant T\}$ with $|p_1|, |p_2|, |p| \gg X, T$, the first and the second harmonic solitons are approximately waves of the form

$$\frac{\exp(2\mathrm{i}\xi_k(\xi_k t+\gamma_k+x))}{p_k} + \mathrm{O}\left(\frac{1}{p_k}\right) \qquad k = 1, 2$$

and, according to (8), their nonlinear superposition is also an oscillatory wave which is the sum of the original harmonic solitons modulus lower-order terms, almost like the linear case.

This phenomenon resembles what is known in physics as interference, and was first discussed in detail for KdV in [Kov2]. Just like the linear case, it should allow us to construct miscellaneous wavepackets or profiles for the defocusing NLS. To do it we need to view (3) as

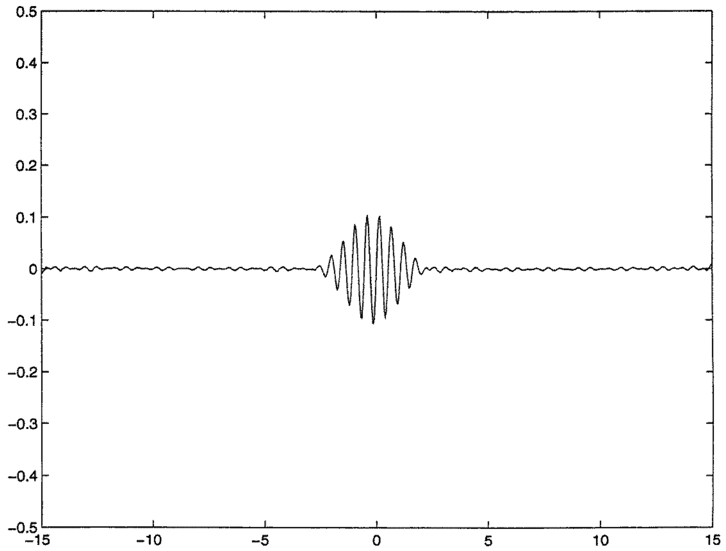

Figure 1. The wavepacket generated by eight harmonic solitons at $t = 0$ with $P = (p_1, \ldots, p_n)$, $p_n = -68\,000 \exp(3(\lambda_n - \lambda_{avg})^2)$, $\Gamma = (0, \ldots, 0)$, $\Lambda = (\lambda_1, \ldots, \lambda_8)$ and $\lambda_n = 3 + 0.056(n-1)$.

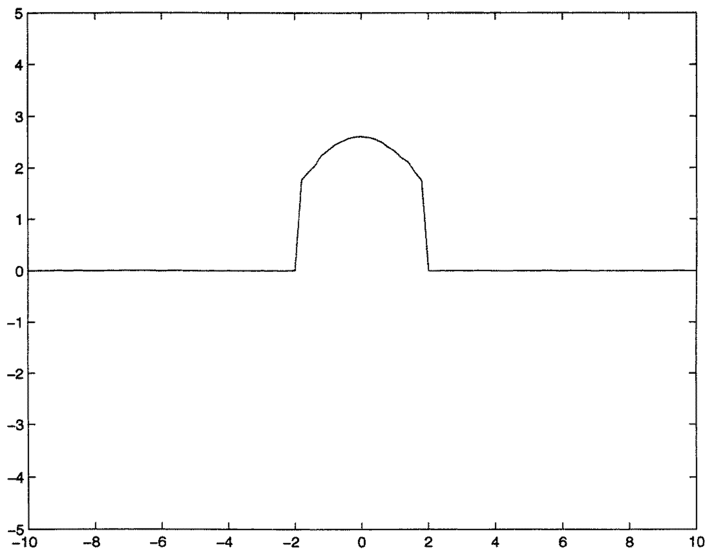

Figure 2. The 13-harmonic soliton solution at $t = 0$ with $P = (p_1, \ldots, p_{13})$, $p_n = -2.8n^2 \times 10^7$, $\Gamma = (0, \ldots, 0)$ and $\lambda = 0.17n$.

a nonlinear analogue of the Fourier integral with p_k playing the role of the Fourier transform. This analogy was employed in [Bar1] to construct a number of wavepackets. Just like the linear case, the wavepackets disperse with time, yet for $t = 0$ they may be constructed to have rather sharp profiles, three of which are given in the figures. Unlike the linear case when wavepackets vanish or almost vanish everywhere outside of a certain interval, the wavepackets for the defocusing NLS vanish or almost vanish outside of a certain interval *but within a much larger interval* which we call the interval of modulation.

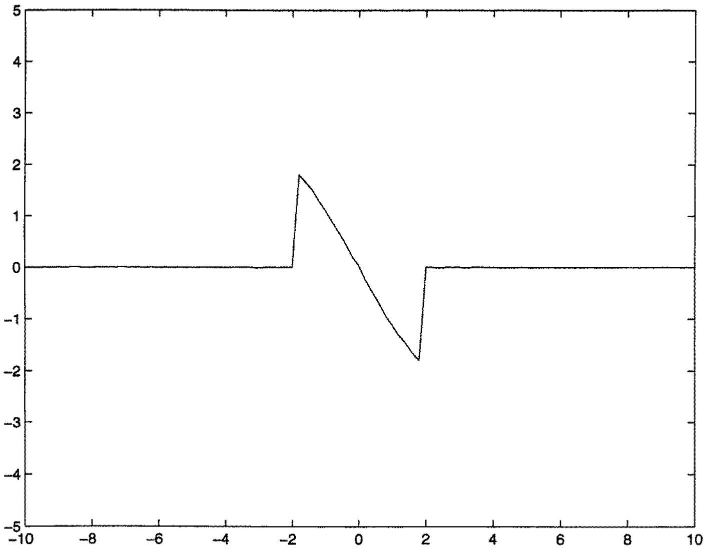

Figure 3. The 13-harmonic soliton solution at $t = 0$ with $P = (p_1, \ldots, p_{13})$, $p_n = -2.8n^2 \times 10^7$, $\Gamma = (\frac{\pi}{2}, \ldots, \frac{\pi}{2})$ and $\lambda = 0.17n$.

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